

HYDRODYNAMIC LIMIT FOR THE GINZBURG-LANDAU $\nabla\phi$ INTERFACE MODEL WITH NON-CONVEX POTENTIAL

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ABSTRACT. Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model was established in [12] under the Dirichlet boundary conditions. This paper studies the similar problem, but with non-convex potentials. Because of the lack of strict convexity, a lot of difficulties arise, especially, on the identification of equilibrium states. We give a proof of the equivalence between the stationarity and the Gibbs property under quite general settings, and as its conclusion, we complete the identification of equilibrium states under the high temperature regime in [2]. We also establish some uniform estimates for variances of extremal Gibbs measures under quite general settings.

1. INTRODUCTION

We consider the large scale hydrodynamic behavior of the the Ginzburg-Landau $\nabla\phi$ interface model. This is an effective interface model, describing the stochastic dynamic of the separation of two distinct phases.

The position of the interface is described by height variables $\phi = \{\phi(x) \in \mathbb{R}; x \in \Gamma\}$ measured from a fixed d -dimensional discrete hyperplane Γ . Here, we will take $\Gamma = \Gamma_N := (\mathbb{Z}/N\mathbb{Z})^d$ when we consider the system on a discretized torus with the periodic boundary condition, or $\Gamma = D_N \subseteq \mathbb{Z}^d$ when we consider the system on the domain Γ with Dirichlet boundary condition. D_N is a microscopic domain corresponding to a given macroscopic domain $D \subset \mathbb{R}^d$ which is bounded and has a smooth boundary. See Section 2 for the precise definition.

The corresponding Hamiltonian $H(\phi)$ on Γ for given height variable ϕ is of the form

$$H(\phi) = \frac{1}{2} \sum_{\substack{x,y \in \Gamma, \\ |x-y|=1}} V(\phi(x) - \phi(y)) + \sum_{\substack{x \in \Gamma, y \in \mathbb{Z}^d \setminus \Gamma, \\ |x-y|=1}} V(\phi(x) - \phi(y)),$$

with a symmetric function $V \in C^2(\mathbb{R})$. The Langevin equation associated with H is given by

$$d\phi_t(x) = -U_x(\phi_t) dt + dw_t(x), \quad x \in \Gamma,$$

where $U_x(\phi)$ in the drift term is defined by

$$U_x(\phi) := \frac{\partial H}{\partial \phi(x)}(\phi) \equiv \sum_{y \in \mathbb{Z}^d; |x-y|=1} V'(\phi(x) - \phi(y))$$

and $\{w_t(x); x \in \Gamma\}$ is a family of independent copies of the one dimensional standard Brownian motion.

The aim of this paper investigate and identify the hydrodynamic limit of ϕ_t at diffusive scaling, that is, N^2 for time while N for space. In the case of a strictly convex potential

V for which there exist two constants $c_+, c_- > 0$ such that

$$c_- \leq V''(\eta) \leq c_+, \quad \eta \in \mathbb{R}. \quad (1.1)$$

the hydrodynamic limit has been established for periodic lattice Γ_N in [7] and for discretized domain D_N with Dirichlet boundary conditions in [12]. In particular, the corresponding macroscopic motion is identified as the solution of the nonlinear partial differential equation

$$\frac{\partial h}{\partial t} = \operatorname{div} \{ (\nabla \sigma)(\nabla h(t, \theta)) \}, \quad \theta \in D, t > 0,$$

where the surface tension $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined via thermodynamic limit.

In these results, the condition (1.1) plays an essential role in the analysis for the stochastic dynamics ϕ_t , especially, in the identification of equilibrium states and the establishment of the strict convexity of σ . The our aim in this paper is to prove the hydrodynamic limit without the strict convexity assumption (1.1), see Assumptions 2.1, 2.2 and 2.3 for details.

Our motivation comes from recent results in [2] and [3] where both strict convexity of the surface tension and identification of the extremal gradient Gibbs measures hold, for non-convex potential V at sufficiently high temperature.

In the case of the dynamics on the torus Γ_N , the limit follows quite simply from additional estimates. However, for the dynamics on the discretized domain D_N with the Dirichlet boundary condition, the derivation is much harder, since we can not use the relative entropy and entropy production. The main step then is to characterize the set of stationary measures for the gradient field associated with the infinite system of SDEs, which is essentially used in order to establish local equilibrium as in [12] without using the relative entropy and the entropy production.

In case of strictly convex V , the structure of the translation invariant stationary measures is completely identified by [7], its proof relying the assumption (1.1). To complete our proof of the hydrodynamic limit in the non-convex case, we need to identify the class of translation invariant stationary measures as the class of Gibbs distributions.

This subject has been intensively studied in the literature, cf. [9] for stochastic Ising models, [10] for the diffusion process on the infinite dimensional torus $(\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^d}$, [5] for the diffusion process on $\mathbb{R}^{\mathbb{Z}^d}$, [13] for the diffusion process on the infinite product $M^{\mathbb{Z}^d}$ with a Riemannian manifold M with positive curvature. In this paper we show the similar result, adapting the argument of [5] to gradient Gibbs distributions. The main challenge here is the lack of ellipticity of the gradient dynamic, see Section 3 and 5 for details.

An alternative derivation of the hydrodynamic limit for the Ginzburg-Landau model based on a two scale argument has been proposed by [8] and [6]. Unlike our proof, relying on the assumption on the uniqueness of the extremal gradient Gibbs distribution, the two scale argument uses logarithmic Sobolev inequalities. However, this approach seems restricted to the one-dimensional case in [8], respectively strict convexity assumption for the potential (1.1) in [6].

Before closing this section, let us give briefly the organization of this paper. In Section 2, we formulate our problem more precisely, and state the main result. In Section 3, we present some properties of translation invariant stationary measures, especially, the relationship between stationarity and the Gibbs property, and some uniform estimates

for their variances. Note that results in this section hold under the quite general Assumption 2.1. In Section 4, after establishing a priori bounds for stochastic dynamics and summarize properties of the surface tension, we derive the macroscopic equation from the stochastic dynamics. Here, we rely quite explicitly on the further Assumptions 2.2 and 2.3. In Section 5, we give a proof of Theorem 3.1, presented at Section 3.

2. MODEL AND MAIN RESULT

2.1. Model. Let D be a bounded domain in \mathbb{R}^d with a Lipschitz boundary. For convenience, let D contain the origin of \mathbb{R}^d . Let D_N be the discretized microscopic domain corresponding to D in the sense that

$$D_N = \{x \in \mathbb{Z}^d; B(x/N, 5/N) \subset D\},$$

where $B(\alpha, l)$ stands for the hypercube in \mathbb{R}^d with center α and side length l , that is,

$$B(\alpha, l) = \prod_{i=1}^d [\alpha_i - l/2, \alpha_i + l/2).$$

On D_N we consider the dynamics governed by the following stochastic differential equations (SDEs)

$$d\phi_t(x) = -U_x(\phi_t) dt + \sqrt{2}dw_t(x), \quad x \in D_N, \quad (2.1)$$

with the boundary condition

$$\phi_t(x) = \psi^N(x), \quad x \in \mathbb{Z}^d \setminus D_N \quad (2.2)$$

with some $\psi^N \in \mathbb{R}^{\mathbb{Z}^d}$ and initial data ϕ_0 , where $U_x(\phi) = \frac{\partial H}{\partial \phi(x)}(\phi)$ for $\phi \in \mathbb{R}^{D_N}$ and $x \in D_N$, or more generally for $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$. The height variable ψ^N in (2.2) is defined by

$$\psi^N(x) = N^{d+1} \int_{B(x/N, 1/N)} f(\theta) d\theta \quad (2.3)$$

for every $x \in \mathbb{Z}^d$, where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a function belonging to $C_0^2(\mathbb{R}^d)$. We note that the function f describes the macroscopic boundary condition and the height variable ψ^N describes the microscopic one.

We make the following assumption on the interaction potential V :

Assumption 2.1. The function $V : \mathbb{R} \rightarrow \mathbb{R}$ has the following representation:

$$V(\eta) = V_0(\eta) + g(\eta), \quad \eta \in \mathbb{R},$$

where functions $V_0, g \in C^2(\mathbb{R})$ are symmetric functions and satisfy

- (1) There exist constants $c_+, c_- > 0$ such that

$$c_- \leq V_0''(\eta) \leq c_+, \quad \eta \in \mathbb{R}.$$

- (2) There exists a constant $C_g > 0$ such that

$$|g'(\eta)| + |g''(\eta)| \leq C_g, \quad \eta \in \mathbb{R}.$$

Example 2.1. If a function $V \in C^2(\mathbb{R})$ is symmetric and satisfies

$$c \leq V''(\eta) \leq c', \quad |x| \geq M$$

for some $c, c' > 0$ and $M > 0$, then the function V admits the decomposition as in Assumption 2.1. Indeed, we can take V_0 as follows:

$$V_0(x) = \begin{cases} \frac{1}{2}V''(M)x^2 - \frac{1}{2}V''(M)M^2 + V(M) + \alpha M, & |x| \leq M, \\ V(x) + \alpha|x|, & |x| > M, \end{cases}$$

with $\alpha = V''(M)M - V'(M)$. Letting $g := V - V_0$, that is,

$$g(x) = \begin{cases} V(x) - V(M) - \frac{1}{2}V''(M)x^2 + \frac{1}{2}V''(M)M^2 - \alpha M, & |x| \leq M, \\ -\alpha|x|, & |x| > M, \end{cases}$$

we can easily see that $V_0, g \in C^2(\mathbb{R})$ and they fulfill conditions (1) and (2) in Assumption 2.1.

Further assumptions dealing with the strict convexity of the surface tension and the characterization of extremal gradient Gibbs measures are stated below, see Assumptions 2.2 and 2.3 for details.

We regard (2.1) as the model describing the motion of microscopic interfaces and introduce the macroscopic height variable h^N as follows:

$$h^N(t, \theta) = \sum_{x \in \mathbb{Z}^d} N^{-1} \phi_{N^2 t}(x) 1_{B(x/N, 1/N)}(\theta), \quad \theta \in \mathbb{R}^d,$$

where $\phi_t = \{\phi_t(x); x \in \mathbb{Z}^d\}$ being the solution of (2.1) with (2.2).

2.2. Notations. Before stating the detail of our main result, we need to introduce several notations. Note that we will follow the same manner as in [7] and [12].

Let $(\mathbb{Z}^d)^*$ be the set of all directed bonds $b = (x, y)$, $x, y \in \mathbb{Z}^d$, $|x - y| = 1$ in \mathbb{Z}^d . We write $x_b = x$ and $y_b = y$ for $b = (x, y)$. We denote the bond $(e_i, 0)$ by e_i again if it doesn't cause any confusion. For every subset Λ of \mathbb{Z}^d , we denote the set of all directed bonds included Λ and touching Λ by Λ^* and $\overline{\Lambda}^*$, respectively. That is,

$$\begin{aligned} \Lambda^* &:= \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda \text{ and } y_b \in \Lambda\}, \\ \overline{\Lambda}^* &:= \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda \text{ or } y_b \in \Lambda\}. \end{aligned}$$

For $\phi = \{\phi(x); x \in \mathbb{Z}^d\} \in \mathbb{R}^{\mathbb{Z}^d}$, the gradient ∇ is defined by

$$\nabla \phi(b) := \phi(x) - \phi(y), \quad b = (x, y) \in (\mathbb{Z}^d)^*.$$

Now, let \mathcal{X} be the family of all gradient fields $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ which satisfy the plaquette condition (2.1) in [7], i.e., $\mathcal{X} = \{\eta \equiv \nabla \phi; \phi \in \mathbb{R}^{\mathbb{Z}^d}\}$. Let \mathbb{L}_r^2 be the set of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ such that

$$|\eta|_r^2 := \sum_{b \in (\mathbb{Z}^d)^*} |\eta(b)|^2 e^{-2r|x_b|} < \infty.$$

We denote $\mathcal{X}_r = \mathcal{X} \cap \mathbb{L}_r^2$ equipped with the norm $|\cdot|_r$. We introduce the dynamics $\eta_t \in \mathcal{X}$ governed by the SDEs

$$d\eta_t(b) = -\nabla U(\eta_t)(b) dt + \sqrt{2} d\nabla w_t(b), \quad b \in (\mathbb{Z}^d)^*, \quad (2.4)$$

where $\{w_t(x); x \in \mathbb{Z}^d\}$ is the family of independent one dimensional Brownian motions. Since the coefficients are Lipschitz continuous in \mathcal{X}_r , this equation has the unique strong solution in \mathcal{X}_r for every $r > 0$. Note that $\eta_t := \nabla \phi_t$ defined from the solution ϕ_t of the SDE (2.1) on D_N satisfies (2.4) for $b \in \overline{D_N^*}$ and boundary conditions $\eta_t(b) = \nabla \psi^N(b)$ for $b \in (\mathbb{Z}^d)^* \setminus \overline{D_N^*}$ when letting $w_t(x) \equiv 0$ for $x \in \mathbb{Z}^d \setminus D_N$.

Since we define Gibbs measures on \mathcal{X} by Dobrushin-Lanford-Ruelle (DLR, for short) equation, we the finite volume Gibbs measure in advance. For a finite set $\Lambda \subset \mathbb{Z}^d$ and fixed $\xi \in \mathcal{X}$, we define the affine space $\mathcal{X}_{\Lambda, \xi} \subset \mathcal{X}$ by

$$\mathcal{X}_{\Lambda, \xi} = \{\eta \in \mathcal{X}; \eta(b) = \xi(b), b \in (\mathbb{Z}^d)^* \setminus \overline{\Lambda^*}\}.$$

We define the finite volume Gibbs measure $\mu_{\Lambda, \xi}$ on $\overline{\Lambda^*}$ by

$$\mu_{\Lambda, \xi}(d\eta) = Z_{\Lambda, \xi}^{-1} \exp \left(- \sum_{b \in \overline{\Lambda^*}} V(\eta(b)) \right) d\eta_{\overline{\Lambda^*}, \xi},$$

where $d\eta_{\overline{\Lambda^*}, \xi}$ is the Lebesgue measure on $\mathcal{X}_{\overline{\Lambda^*}, \xi}$ and $Z_{\Lambda, \xi}$ is the normalizing constant.

Let $\mathcal{P}(\mathcal{X})$ be the set of all probability measures on \mathcal{X} and let $\mathcal{P}_2(\mathcal{X})$ be those $\mu \in \mathcal{P}(\mathcal{X})$ satisfying $E^\mu[|\eta(b)|^2] < \infty$ for each $b \in (\mathbb{Z}^d)^*$. The measure $\mu \in \mathcal{P}_2(\mathcal{X})$ is sometimes called tempered. Let \mathcal{G} be the family of translation invariant, tempered Gibbs measures $\mu \in \mathcal{P}_2(\mathcal{X})$ introduced by [7], namely, the family of $\mu \in \mathcal{P}_2(\mathcal{X})$ satisfying the Dobrushin-Lanford-Ruelle equation

$$\mu(\cdot | \mathcal{F}_{(\mathbb{Z}^d)^* \setminus \overline{\Lambda^*}}) = \mu_{\Lambda, \xi}(\cdot), \quad \mu\text{-a.s. } \xi, \quad (2.5)$$

where $\mathcal{F}_{(\mathbb{Z}^d)^* \setminus \overline{\Lambda^*}}$ is the σ -algebra generated by $\{\eta(b); b \in (\mathbb{Z}^d)^* \setminus \overline{\Lambda^*}\}$. Note that the dynamics η_t given by (2.4) is reversible under $\mu \in \mathcal{G}$. We denote the family of $\mu \in \mathcal{G}$ with ergodicity under spatial shifts by \mathcal{G}_{ext} .

2.3. Assumptions on Gibbs measures and the surface tension. In order to derive the hydrodynamic limit, we will assume both uniqueness of the extremal gradient Gibbs distributions and strict convexity of the surface tension. These assumption are always satisfied under (1.1), cf. see [4] and [7], or for non-convex potential V at sufficiently high temperature, cf. [2] and [3]. On the other hand, at critical temperature, Biskup and Kotecký give an example of gradient Gibbs measures with two different extremal states, cf. [1]. The derivation of the corresponding hydrodynamic limit in this case is very challenging open problem.

More precisely, let $\Gamma_N, N \in \mathbb{N}$ be the periodic lattice $(\mathbb{Z}/N\mathbb{Z})^d$ and Γ_N^* be the set of all directed bonds in Γ_N . With $\mathcal{X}_{\Gamma_N} = \{\nabla \phi \in \mathbb{R}^{\Gamma_N^*}; \phi \in \mathbb{R}^{\Gamma_N}\}$, we consider the finite volume Gibbs measure $\tilde{\mu}_{N, u}$ on \mathcal{X}_{Γ_N} by

$$\tilde{\mu}_{N, u}(d\tilde{\eta}) = Z_{N, u}^{-1} \exp \left(- \frac{1}{2} \sum_{b \in \Gamma_N^*} V(\tilde{\eta}(b) + u_b) \right) d\tilde{\eta},$$

where $d\tilde{\eta}$ is Lebesgue measure on \mathcal{X}_{Γ_N} , $Z_{N,u}$ is the normalizing constant and u_b is defined by $u_b = \pm u_i$ for $b = (x \pm e_i, x)$ with $x \in \Gamma_N$ and $1 \leq i \leq d$. We denote the law of $\{\eta(b) + u_b\}$ by $\mu_{N,u}$.

Assumption 2.2. For each $u \in \mathbb{R}^d$ there exists a unique extremal $\mu_u \in \mathcal{G}_{\text{ext}}$ such that

$$E^{\mu_u}[\eta(e_i)] = u_i.$$

Furthermore, it can be obtained as the weak limit of the periodic Gibbs $\mu_{N,u}$ as $N \rightarrow \infty$.

Under Assumption 2.2, the sequence $\{\sigma_N(u)\}$ defined by

$$\sigma_N(u) := -|\Gamma_N|^{-1} (\log Z_{N,u} - \log Z_{N,0}),$$

has a limit. We thus define the (normalized) surface tension $\sigma(u)$, $u \in \mathbb{R}^d$ by

$$\sigma(u) = \lim_{N \rightarrow \infty} \sigma_N(u). \quad (2.6)$$

Moreover, we can show the following thermodynamic identities between the surface tension and ergodic Gibbs measures:

$$E^{\mu_u}[V'(\eta(e_i))] = \nabla \sigma(u), \quad u \in \mathbb{R}^d, \quad (2.7)$$

$$E^{\mu_u} \left[\sum_{i=1}^d \eta(e_i) V'(\eta(e_i)) \right] = u \cdot \nabla \sigma(u) + 1, \quad u \in \mathbb{R}^d, \quad (2.8)$$

which will be shown in Section 4.2. They play an essential role in the derivation of the hydrodynamic limit.

Further we need some technical assumption on the regularity of σ which are well known in the strictly convex case (1.1), cf. [7] or in the high temperature regime [2].

Assumption 2.3. The surface tension σ is C^1 and $\nabla \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz continuous. Furthermore, σ is strictly convex in the following sense: there exist two constants $C_1, C_2 > 0$ satisfying

$$C_1 |u - v|^2 \leq (u - v) \cdot (\nabla \sigma(u) - \nabla \sigma(v)) \leq C_2 |u - v|^2, \quad u, v \in \mathbb{R}^d. \quad (2.9)$$

Remark 2.1. Note that the convexity of the surface tension, alternatively defined in terms of fixed boundary conditions has been established in [11] under very general conditions. Moreover, the strict convexity (i.e. lower bound in (2.9) with $C_1 > 0$) is not essential for the hydrodynamic limit since an approximation of σ could be implemented as in [7].

The following example shows that our Assumptions 2.2 and 2.3 hold in the high temperature regime:

Example 2.2. We introduce a positive parameter $\beta > 0$ corresponding to the inverse temperature, that is, the potential V takes the form

$$V(\eta) = \beta(\tilde{V}_0(\eta) + \tilde{g}(\eta)),$$

where the symmetric functions $\tilde{V}_0, \tilde{g} \in C^2(\mathbb{R})$ satisfy

$$0 < c_- \leq \tilde{V}_0'' \leq c_+ < \infty \quad -\infty < -d_- < \tilde{g}'' \leq d_+ < \infty$$

for some $c_- < d_-$ and $\|g''\|_{L^q(\mathbb{R})} < \infty$ for some $q \geq 1$. Then for $\beta_0 = \beta_0(c_-, c_+ + d_+, \|g''\|_{L^q(\mathbb{R})}) > 0$, (independent of $d_-!$) of the form

$$\beta_0 = \frac{(c_-)^{3q}}{2d \, 2^{2q} (c_+ + d_+)^{q+1} \|g''\|_{L^q(\mathbb{R})}^{2q}}$$

both Assumptions 2.2 and 2.3 are satisfied when $\beta \leq \beta_0$, see [2] and its arXiv version (arXiv:0807.2621v1 [math.PR]).

2.4. Main Result. The main result in this paper is the following:

Theorem 2.1. *We assume Assumptions 2.1, 2.2 and 2.3. Furthermore, we assume that there exists $h_0 \in C^2(D)$ satisfying the following:*

- (1) *The function $h_0 - f$ has a compact support in D .*
- (2) *The sequence of initial data $\phi_0 = \phi_0^N$ for (2.1) satisfies*

$$\lim_{N \rightarrow \infty} E \|h^N(0) - h_0\|_{L^2(D)}^2 = 0, \quad (2.10)$$

where $h^N(0)$ is the macroscopic height variable corresponding to ϕ_0^N .

Then, for every $t > 0$, $h^N(t)$ converges in L^2 as $N \rightarrow \infty$ to $h(t)$ which is the unique weak solution of the partial differential equation (PDE)

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} h(t, \theta) = \operatorname{div} \left\{ (\nabla \sigma)(\nabla h(t, \theta)) \right\} \\ \equiv \sum_{i=1}^d \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \sigma}{\partial u_i} (\nabla h(t, \theta)) \right\}, \quad \theta \in D, t > 0 \\ h(t, \theta) = f(\theta), \quad \theta \in D^c, t \geq 0 \\ h(0, \theta) = h_0(\theta), \quad \theta \in D, \end{array} \right. \quad (2.11)$$

where $\nabla h = (\partial h / \partial \theta_i)_{i=1}^d$. Here, the function $\sigma = \sigma(u)$ is the surface tension. More precisely, for every $t > 0$,

$$\lim_{N \rightarrow \infty} E \|h^N(t) - h(t)\|_{L^2(D)}^2 = 0 \quad (2.12)$$

holds.

3. STATIONARY MEASURES AND ESTIMATE FOR VARIANCE

In this section, we mainly discuss properties of stationary measures of (2.4) while working on the general assumption, Assumption 2.1. We believe that the results of this section are relevant beyond the derivation of the hydrodynamic limit.

3.1. Generator of (2.4) and stationary measures. We at first note that the infinitesimal generator of (2.4) is given by

$$\mathcal{L}^{\mathbb{Z}^d} = \sum_{x \in \mathbb{Z}^d} \mathcal{L}_x, \quad (3.1)$$

where

$$\mathcal{L}_x = \sum_{b, b' \in (\mathbb{Z}^d)^* : x_b = x_{b'} = x} \left\{ 4 \frac{\partial^2}{\partial \eta(b) \partial \eta(b')} - 2V'(\eta(b)) \frac{\partial}{\partial \eta(b')} \right\}.$$

To keep notation simple, we sometimes denote $\mathcal{L}^{\mathbb{Z}^d}$ by \mathcal{L} if it doesn't cause any confusion.

We can see that the Gibbs property implies reversibility under (2.4), and therefore stationarity, see Proposition 3.1 in [7] for details. We note that the same argument as in [7] is applicable in quite general setting, including ours. In Theorem 2.1 of [7], the equivalence of the Gibbs property and stationarity is shown using (1.1), here we show this result using another approach.

Theorem 3.1. *We assume Assumption 2.1. If $\mu \in \mathcal{P}_2(\mathcal{X})$ is invariant under spatial shift and a stationary measure corresponding to \mathcal{L} , i.e.,*

$$\int_{\mathcal{X}} \mathcal{L}f(\eta)\mu(d\eta) = 0, \quad f \in C_{\text{loc}}^2(\mathcal{X}),$$

then μ is a Gibbs measure, i.e., (2.5) holds.

Since the proof of Theorem 3.1 is slightly long, we postpone the proof until the end of this paper, see Section 5.

3.2. Uniform bound for the variance for stationary measures. If the potential V is a strictly convex function satisfying (1.1), we then get the uniform bound for the variance for Gibbs measures as a direct consequence of the Brascamp-Lieb inequality. See [4] for details. Our next result based on dynamical approach shows that the variance remains bounded in the tilt u for general potentials under Assumption 2.1.

Theorem 3.2. *We assume Assumption 2.1. Let \mathcal{S}_{ext} be the family of stationary measures for the gradient field (2.4) which are tempered, translation invariant and ergodic under spatial shift. The variance of $\eta(b)$, $b \in (\mathbb{Z}^d)^*$ under μ are bounded from above by a constant independent of μ , that is,*

$$\sup_{\mu \in \mathcal{S}_{\text{ext}}} \text{Var}_{\mu}[\eta(b)] < \infty, \quad b \in (\mathbb{Z}^d)^*$$

holds.

Proof. We shall show the desired bound by arranging the argument of the proof of Proposition 2.1 of [7]. We fix $\mu \in \mathcal{S}_{\text{ext}}$ and we define the vector $u = (u_i)_{1 \leq i \leq d} \in \mathbb{R}^d$ by

$$u_i = E^{\mu}[\eta((e_i, 0))], \quad 1 \leq i \leq d.$$

Let $\eta_t \in \mathcal{X}$ be the solution of SDEs (2.4) with initial distribution μ . Introducing $\phi_t \in \mathbb{R}^{\mathbb{Z}^d}$ by

$$\phi_t(0) = \int_0^t U_0(\eta_s) ds + \sqrt{2}w_t(0)$$

and

$$\phi_t(x) = \phi_t(0) + \sum_{b \in C_{0,x}} \eta_t(b), \quad x \in \mathbb{Z}^d,$$

where $C_{0,x}$ is an arbitrary chain connecting 0 to x , we then obtain that ϕ_t solves the SDEs

$$d\phi_t(x) = -U_x(\phi_t) dt + \sqrt{2}dw_t(x), \quad x \in \mathbb{Z}^d.$$

Our calculation will be based on the energy estimate for ϕ_t introduced above.

Let $\ell \geq 1$ and $\Lambda \equiv \Lambda_\ell = [-\ell, \ell]^d \cap \mathbb{Z}^d$. For a deterministic $\psi \in \mathbb{R}^{\mathbb{Z}^d}$ with

$$\psi(x) = u \cdot x, \quad x \in \mathbb{Z}^d,$$

we obtain

$$d \sum_{x \in \Lambda} (\phi_t(x) - \psi(x))^2 = -2 \sum_{x \in \Lambda} (\phi_t(x) - \psi(x)) U_x(\phi_t) dt + 2|\Lambda| dt + M_t$$

with a martingale M_t by Itô's formula. Performing summation-by-parts, we get

$$\begin{aligned} \sum_{x \in \Lambda} (\phi_t(x) - \psi(x)) U_x(\phi_t) &= \frac{1}{2} \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b)) V'(\nabla \phi_t(b)) \\ &\quad - \sum_{b \in \overline{\Lambda^*}; x_b \in \Lambda^c} (\phi_t(x_b) - \psi(x_b)) V'(\nabla \phi_t(b)). \end{aligned}$$

We thus have

$$\sum_{x \in \Lambda} (\phi_T(x) - \psi(x))^2 = I_0 + I_1(T) + I_2(T) + 2|\Lambda|T + M_T, \quad (3.2)$$

where $I_0, I_1(T)$ and $I_2(T)$ are defined by

$$\begin{aligned} I_0 &= \sum_{x \in \Lambda} (\phi_0(x) - \psi(x))^2, \\ I_1(T) &= - \int_0^T \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b)) V'(\nabla \phi_t(b)) dt, \\ I_2(T) &= 2 \int_0^T \sum_{b \in \overline{\Lambda^*}; x_b \in \Lambda^c} (\phi_t(x_b) - \psi(x_b)) V'(\nabla \phi_t(b)) dt. \end{aligned}$$

From now on, we shall give bounds for expectations of $I_0, I_1(T)$ and $I_2(T)$ separately.

We at first give a estimate for the expectation of I_0 . Here, the same argument as the proof of (2.14) in [7] can be applied. That is, from ergodicity and temperedness of μ , we have

$$\lim_{|x| \rightarrow \infty} \frac{1}{|x|^2} E[(\phi_0(x) - \psi(x))^2] = 0, \quad (3.3)$$

and this implies that

$$\lim_{\ell \rightarrow \infty} \ell^{-2} |\Lambda|^{-1} E[I_0] = 0.$$

We therefore obtain that for every $\epsilon > 0$ there exists $\ell_0 \geq 1$ such that

$$E[I_0] \leq \epsilon \ell^2 |\Lambda| \quad (3.4)$$

holds for every $\ell \geq \ell_0$.

We shall next calculate $I_1(T)$ and its expectation. From Assumption 2.1, $I_1(T)$ can be calculated as follows:

$$I_1(T) = - \int_0^T \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b)) (V'_0(\nabla \phi_t(b)) - V'_0(\nabla \psi(b))) dt$$

$$\begin{aligned}
& - \int_0^T \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b)) g'(\nabla \phi_t(b)) dt \\
& + \int_0^T \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b)) (V'_0(\nabla \psi(b))) dt \\
& \leq -c_- \int_0^T \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b))^2 dt \\
& + \|g'\|_\infty \int_0^T \sum_{b \in \overline{\Lambda^*}} |\nabla \phi_t(b) - \nabla \psi(b)| dt \\
& + \int_0^T \sum_{b \in \overline{\Lambda^*}} (\nabla \phi_t(b) - \nabla \psi(b)) V'_0(\nabla \psi(b)) dt \\
& =: I_{1,1}(T) + I_{1,2}(T) + I_{1,3}(T).
\end{aligned}$$

Using Schwarz's inequality, we obtain the following estimate for the second term $I_{1,2}(T)$:

$$I_{1,2}(T) \leq \frac{1}{2} \lambda \|g'\|_\infty \int_0^T \sum_{b \in \overline{\Lambda^*}} |\nabla \phi_t(b) - \nabla \psi(b)|^2 dt + \frac{1}{2} \lambda^{-1} \|g'\|_\infty |\overline{\Lambda^*}| T$$

for arbitrary $\lambda > 0$. If $\|g'\|_\infty > 0$ holds, we then have

$$I_{1,2}(T) \leq \frac{1}{2} c_- \int_0^T \sum_{b \in \overline{\Lambda^*}} |\nabla \phi_t(b) - \nabla \psi(b)|^2 dt + \frac{1}{2} c_-^{-1} \|g'\|_\infty^2 |\overline{\Lambda^*}| T \quad (3.5)$$

by taking $\lambda = c_- \|g'\|_\infty^{-1}$. Note that the estimate (3.5) trivially holds when $\|g'\|_\infty = 0$. Summarizing above and taking expectation, we obtain

$$E[I_1(T)] \leq -\frac{1}{2} c_- \int_0^T \sum_{b \in \overline{\Lambda^*}} E[(\nabla \phi_t(b) - \nabla \psi(b))^2] dt + \frac{1}{2} c_-^{-1} \|g'\|_\infty^2 |\overline{\Lambda^*}| T.$$

Here, we have used

$$E[I_{1,3}(T)] = 0,$$

which follows from the definition of ψ and u . From the relationship $\nabla \phi_t = \eta_t$, the stationarity of μ and the definition of u , we have

$$E[(\nabla \phi_t(b) - \nabla \psi(b))^2] = \text{Var}_\mu[\eta(b)].$$

Since μ is translation invariant, we also have

$$\sum_{b \in \overline{\Lambda^*}} \text{Var}_\mu[\eta(b)] \geq \kappa |\Lambda| \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)]$$

with a constant $\kappa > 0$. Applying above, we finally conclude

$$E[I_1(T)] \leq -\frac{1}{2} c_- \kappa T |\Lambda| \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)] + \frac{1}{2} c_-^{-1} \|g'\|_\infty^2 |\overline{\Lambda^*}| T. \quad (3.6)$$

We next calculate the expected value of $I_2(T)$. Putting $\tilde{I}_2(T)$ by

$$\tilde{I}_2(T) = 2 \int_0^T \sum_{b \in \overline{\Lambda}^*; x_b \in \Lambda^{\mathfrak{G}}} (\phi_t(x_b) - \psi(x_b))(V'(\nabla \phi_t(b)) - V'(\nabla \psi(b))) dt,$$

we have

$$E[I_2(T)] = E[\tilde{I}_2(T)]$$

from the definition of u . We shall thus calculate $\tilde{I}_2(T)$ instead of $I_2(T)$. Using Schwarz's inequality, we obtain

$$\begin{aligned} E[\tilde{I}_2(T)] &\leq \gamma \ell^{-1} |\partial \Lambda_\ell^*| \int_0^T \sup_{y \in \partial \Lambda} E[(\phi_t(y) - \psi(y))^2] dt \\ &\quad + \gamma^{-1} \ell \int_0^T \sum_{b \in \overline{\Lambda}^*; x_b \in \Lambda^{\mathfrak{G}}} E[(V'(\nabla \phi_t(b)) - V'(\nabla \psi(b)))^2] dt \\ &=: F_{2,1}(T) + F_{2,2}(T) \end{aligned} \tag{3.7}$$

for an arbitrary $\gamma > 0$, where $\partial \overline{\Lambda}^* \subset (\mathbb{Z}^d)^*$ and $\partial \Lambda$ are define by

$$\begin{aligned} \partial \overline{\Lambda}^* &= \left\{ b \in \overline{\Lambda}^*; x_b \in \Lambda^{\mathfrak{G}} \right\}, \\ \partial \Lambda &= \{ x_b; b \in \partial \overline{\Lambda}^* \}. \end{aligned}$$

For $F_{2,2}(T)$, since V' is Lipschitz continuous, there exists a constant $C > 0$ such that

$$F_{2,2} \leq C \gamma^{-1} \ell^d T \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)] \tag{3.8}$$

by using the translation invariance of μ . For $F_{2,1}(T)$, let us use a similar argument to the proof of (2.12) in [7]. Taking $\Lambda' = \Lambda_{[\ell/2]}$, we have

$$\begin{aligned} (\phi_t(y) - \psi(y))^2 &\leq 2 \left(\phi_t(y) - \psi(y) - \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} (\phi_t(x) - \psi(x)) \right)^2 \\ &\quad + 2 \left(\frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} (\phi_t(x) - \psi(x)) \right)^2 \\ &=: A_1 + A_2 \end{aligned}$$

for every $y \in \partial \Lambda_\ell$. For the term A_1 , the calculations runs quite parallel to the argument in [7] and we can obtain that for every $\epsilon > 0$ there exists $\ell_1 \geq 1$ such that

$$E[A_1] \leq \epsilon \ell^2$$

holds for every $\ell \geq \ell_1$. Let us give a bound for the term A_2 . Using Itô's formula, we obtain

$$\begin{aligned} \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} (\phi_t(x) - \psi(x)) &= \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} (\phi_0(x) - \psi(x)) \\ &\quad - \frac{1}{|\Lambda'|} \int_0^t \sum_{x \in \Lambda'} \sum_{b \in (\mathbb{Z}^d)^*; x_b = x} V'(\eta_s(b)) ds + \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} w_t(x) \end{aligned}$$

and therefore we get

$$\begin{aligned} A_2 &\leq 4 \left(\frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} (\phi_0(x) - \psi(x)) \right)^2 \\ &\quad + 4 \left(\frac{1}{|\Lambda'|} \int_0^t \sum_{x \in \Lambda'} \sum_{b \in (\mathbb{Z}^d)^*; x_b = x} V'(\eta_s(b)) ds \right)^2 + 4 \left(\frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} w_t(x) \right)^2 \\ &=: A_{2,1} + A_{2,2} + A_{2,3}. \end{aligned}$$

Similarly to (3.4), we obtain that for every $\epsilon > 0$ there exists $\ell_2 \geq 1$ such that

$$E[A_{2,1}] \leq \epsilon \ell^2$$

holds for every $\ell \geq \ell_2$. We also obtain

$$E[A_{2,3}] = \frac{2t}{|\Lambda'|}$$

by a simple calculation. We shall estimate the term $A_{2,2}$. We note that

$$\frac{1}{|\Lambda'|} \int_0^t \sum_{x \in \Lambda'} \sum_{b \in (\mathbb{Z}^d)^*; x_b = x} (V'(\eta_s(b)) - V'(\psi(b))) ds = \frac{1}{|\Lambda'|} \int_0^t \sum_{b \in B_\ell} (V'(\eta_s(b)) - V'(\psi(b))) ds,$$

where

$$B_\ell = \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda', y_b \notin \Lambda'\}.$$

Here, we have used

$$\sum_{x \in \Lambda'} \sum_{b \in (\mathbb{Z}^d)^*; x_b = x} V'(\psi(b)) = 0,$$

which follows from the definition of ψ and the symmetry of V . We therefore obtain

$$E[A_{2,2}] \leq \frac{(c_+ + C_g)^2 |B_\ell|^2 t^2}{|\Lambda'|^2} \sum_{b: x_b = 0} \text{Var}_\mu[\eta(b)].$$

Summarizing above, we conclude the following: for every $\epsilon > 0$ there exists $L \geq 1$ such that

$$\sup_{y \in \partial \Lambda_\ell} E[(\phi_t(y) - \psi(y))^2] \leq C' \left(\epsilon \ell^2 + \ell^{-2} t^2 \sum_{b: x_b = 0} \text{Var}_\mu[\eta(b)] + \ell^{-d} t \right) \quad (3.9)$$

for every $t \geq 0$ and $\ell \geq L$ with a constant $C' > 0$. Note that the constant C' does not depend on μ while L may depend on μ . Combining (3.7) with (3.8) and (3.9), we get the following bound for $E[I_2]$:

$$\begin{aligned} E[I_2(T)] &\leq C' \gamma \epsilon^2 \ell T |\partial \bar{\Lambda}^*| + C' \gamma \ell^{-3} T^3 |\partial \bar{\Lambda}^*| \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)] \\ &\quad + C \gamma^{-1} T \ell |\partial \bar{\Lambda}^*| \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)] + C' \gamma |\partial \bar{\Lambda}^*| \ell^{-d-1} T^2 \end{aligned} \quad (3.10)$$

for every $\epsilon > 0$ and ℓ large enough.

Inserting (3.4), (3.6) and (3.10) into the expectation of (3.2) divided by $|\Lambda|T$, we obtain

$$\begin{aligned} &\left(\frac{1}{2} c_- \kappa - C' \gamma |\partial \bar{\Lambda}^*| |\Lambda|^{-1} \ell^{-3} T^2 - C \gamma^{-1} \ell |\Lambda|^{-1} |\partial \bar{\Lambda}^*| \right) \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)] \\ &\leq \epsilon^2 \ell^2 T^{-1} + \frac{1}{2} c_-^{-1} \|g'\|_\infty^2 |\bar{\Lambda}^*| |\Lambda|^{-1} + C' \gamma \epsilon^2 \ell |\partial \bar{\Lambda}^*| |\Lambda|^{-1} \\ &\quad + C' \gamma \ell^{-d-1} T |\partial \bar{\Lambda}^*| |\Lambda|^{-1} \end{aligned}$$

for every $\epsilon > 0$ and ℓ large enough. Here, taking $T = \gamma^{-1} \ell^2$ and recalling the definition of Λ , $\partial \bar{\Lambda}^*$ and $\bar{\Lambda}^*$, we obtain

$$\left(\frac{1}{2} c_- \kappa - C_1 \gamma^{-1} \right) \sum_{b: x_b=0} \text{Var}_\mu[\eta(b)] \leq C_2 \epsilon^2 \gamma + C_3 \quad (3.11)$$

with constants $C_1, C_2, C_3 \geq 0$. We emphasize that constants appearing on (3.11) does not depend on μ . Choosing γ large enough such that

$$\frac{1}{2} c_- \kappa - C_1 \gamma^{-1} > 0,$$

we conclude the desired bound. \square

Remark 3.1. The argument in the proof of Theorem 3.2 can be applied also to the finite volume Gibbs measures defined in Section 2.3. Under Assumption 2.1, the variance of $\eta(b), b \in (\mathbb{Z}^d)^*$ under $\mu_{N,u}$ are bounded from above by a constant independent of u and N , that is,

$$\sup_{N \geq 1} \sup_{u \in \mathbb{R}^d} \text{Var}_{\mu_{N,u}}[\eta(b)] < \infty, \quad b \in (\mathbb{Z}^d)^*$$

holds. The above implies that the sequence $\{\mu_{N,u}; N \geq 1\}$ is tight for given $u \in \mathbb{R}^d$ and every limit point is a tempered, translation invariant Gibbs measure.

4. THE PROOF OF THEOREM 2.1

In this section, we shall complete our main result, Theorem 2.1. To do so, we at first summarize properties of the surface tension σ . The estimate established in the previous sections will play a key role in the proofs. After that, we finish the proof of Theorem 2.1.

4.1. A priori bounds for the macroscopic height variable. We shall derive the bound corresponding to Proposition 4.1 in [12]. Once we have Proposition 4.1 and Theorem 3.1 in Section 3, we can follow the argument of [12] assuming that the limit of the initial datum is smooth enough.

Proposition 4.1. *There exists a constant $K > 0$ depending f and V such that*

$$E \|h^N(t)\|_{L^2(D)}^2 + c_- N^{-d} E \int_0^t \sum_{b \in \overline{D_N^*}} (\nabla \phi_s^N(b))^2 ds \leq 2E \|h^N(0)\|_{L^2(D)}^2 + K(1+t),$$

where $\phi_s^N \in \mathbb{R}^{\mathbb{Z}^d}$ is defined by $\phi_s^N(x) := \phi_{N^2s}(x)$ for $x \in \mathbb{Z}^d$.

Proof. Using Itô's formula, we have

$$\begin{aligned} \|h^N(t) - f^N\|_{L^2(D)}^2 &= N^{-d-2} \sum_{x \in D_N} (\phi_0^N(x) - \psi^N(x))^2 \\ &\quad - 2N^{-d} \int_0^t \sum_{x \in D_N} (\phi_s^N(x) - \psi^N(x)) \partial_x H(\phi_s^N) ds \\ &\quad + 2N^{-d} |D_N| t + M_t^N, \end{aligned}$$

where M_t^N is a martingale. Performing the summation-by-parts at the second term in the right hand side, we obtain

$$\begin{aligned} &-2N^{-d} \int_0^t \sum_{x \in D_N} (\phi_s^N(x) - \psi^N(x)) \partial_x H(\phi_s^N) ds \\ &= -N^{-d} \int_0^t \sum_{b \in \overline{D_N^*}} \nabla \phi_s^N(b) V_0'(\nabla \phi_s^N(b)) ds \\ &\quad + N^{-d} \int_0^t \sum_{b \in \overline{D_N^*}} \nabla \psi^N(b) V_0'(\nabla \phi_s^N(b)) ds \\ &\quad - N^{-d} \int_0^t \sum_{b \in \overline{D_N^*}} (\nabla \phi_s^N(b) - \nabla \psi^N(b)) g'(\nabla \phi_s^N(b)) ds \\ &=: N^{-d} \int_0^t I_1(s) ds + N^{-d} \int_0^t I_2(s) ds + N^{-d} \int_0^t I_3(s) ds. \end{aligned}$$

Here, we have used the boundary condition $\phi_t^N(x) = \psi^N(x)$ for $x \in \mathbb{Z}^d \setminus D_N$ and $t \geq 0$. For the main part $I_1(s)$, we have

$$I_1(s) \leq -c_- \sum_{b \in \overline{D_N^*}} (\nabla \phi_s^N(b))^2$$

from the strict convexity of V_0 . Next, we have for $I_2(s)$

$$|I_2(s)| \leq C_f \sum_{b \in \overline{D_N^*}} |V_0'(\nabla \phi_s^N(b))|$$

$$\begin{aligned}
&\leq C_f c_+ \sum_{b \in \overline{D_N^*}} |\nabla \phi_s^N(b)| \\
&\leq \frac{1}{4} c_- \sum_{b \in \overline{D_N^*}} (\nabla \phi_s^N(b))^2 + 4C_f^2 c_+^2 c_-^{-1} |\overline{D_N^*}|,
\end{aligned}$$

where the constant C_f is defined by

$$C_f := \sup_{1 \leq i, j \leq d} \sup_{\theta \in \mathbb{R}^d} \left| \frac{\partial f}{\partial \theta_i}(\theta) \right|.$$

Finally, for $I_3(s)$, we have

$$\begin{aligned}
|I_3(s)| &\leq C_g \sum_{b \in \overline{D_N^*}} |\nabla \phi_s^N(b) - \nabla \psi^N(b)| \\
&\leq 2\gamma C_g \sum_{b \in \overline{D_N^*}} |\nabla \phi_s^N(b) - \nabla \psi^N(b)|^2 + 2\gamma^{-1} C_g |\overline{D_N^*}| \\
&\leq 4\gamma C_g \sum_{b \in \overline{D_N^*}} |\nabla \phi_s^N(b)|^2 + 4\gamma C_g \sum_{b \in \overline{D_N^*}} |\nabla \psi^N(b)|^2 + 2\gamma^{-1} C_g |\overline{D_N^*}|
\end{aligned}$$

for an arbitrary $\gamma > 0$. Choosing $\gamma = c_-(16C_g)^{-1}$, we have

$$\begin{aligned}
&\left| \sum_{b \in \overline{D_N^*}} (\nabla \phi_s^N(b) - \nabla \psi^N(b)) g'(\nabla \phi_s^N(b)) \right| \\
&\leq \frac{1}{4} c_- \sum_{b \in \overline{D_N^*}} |\nabla \phi_s^N(b)|^2 + \left(\frac{1}{4} c_- C_f^2 + 32C_g^2 c_-^{-1} \right) |\overline{D_N^*}|.
\end{aligned}$$

Summarizing above, we get

$$\begin{aligned}
\|h^N(t) - f^N\|_{L^2(D)}^2 &\leq \|h^N(0) - f^N\|_{L^2(D)}^2 - \frac{1}{2} c_- N^{-d} \int_0^t \sum_{b \in \overline{D_N^*}} (\nabla \phi_s^N(b))^2 ds \\
&\quad + \left(4C_f^2 c_+^2 c_-^{-1} + \frac{1}{4} c_- C_f^2 + 32C_g^2 c_-^{-1} \right) N^{-d} |\overline{D_N^*}| t \\
&\quad + 2N^{-d} |D_N| t + M_t^N.
\end{aligned}$$

Taking the expectation, we obtain the conclusion. \square

4.2. Surface tension and thermodynamic identities. In this subsection, we verify several properties of surface tension σ . Note that in view of our estimate of the variance, Theorem 3.2, we easily can get the identity (2.7) following the argument of [7]. The proof of the second equality (2.8) is more delicate, since, in view of the missing higher moment estimate, we cannot immediately apply the argument of [7].

Proposition 4.2. *For every translation-invariant, ergodic Gibbs measure μ_u , we have*

$$E^{\mu_u} \left[\sum_{i=1}^d \eta(e_i) V'(\eta(e_i)) \right] = u \cdot \nabla \sigma(u) + 1.$$

Proof. Let $\ell \geq 1$ and we denote Λ_ℓ simply by Λ . We define $\psi \in \mathbb{R}^{\mathbb{Z}^d}$ by $\psi(x) = u \cdot x$ for $x \in \mathbb{Z}^d$. We at first note that

$$\begin{aligned} \sum_{b \in \overline{\Lambda}^*} (\eta(b) - \nabla \psi(b)) V'(\eta(b)) &= 2 \sum_{x \in \Lambda} (\phi^{0,\eta}(x) - \psi(x)) \frac{\partial H}{\partial \phi(x)}(\phi^{0,\eta}) \\ &\quad - 2 \sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} (\phi^{0,\eta}(x_b) - \psi(x_b)) V'(\eta(b)) \end{aligned}$$

holds by the summation-by-parts. Since we have

$$\begin{aligned} E^{\mu_u} \left[2 \sum_{x \in \Lambda} (\phi^{0,\eta}(x) - \psi(x)) \frac{\partial H}{\partial \phi(x)}(\phi^{0,\eta}) \right] \\ = E^{\mu_u} \left[E^{\mu_{\Lambda, \varepsilon}} \left[2 \sum_{x \in \Lambda} (\phi^{0,\eta}(x) - \psi(x)) \frac{\partial H}{\partial \phi(x)}(\phi^{0,\eta}) \right] \right] \\ = 2|\Lambda| - 2 \end{aligned}$$

from the DLR equation and the integration-by-parts, we obtain

$$\begin{aligned} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*} (\eta(b) - \nabla \psi(b)) V'(\eta(b)) \right] \\ = 2|\Lambda| - 2 - 2E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} (\phi^{0,\eta}(x_b) - \psi(x_b)) V'(\eta(b)) \right]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*} (\eta(b) - \nabla \psi(b)) V'(\eta(b)) \right] \\ = E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*} \eta(b) V'(\eta(b)) \right] - \sum_{b \in \overline{\Lambda}^*} \nabla \psi(b) E^{\mu_u} [V'(\eta(b))], \end{aligned}$$

and therefore we obtain

$$\begin{aligned} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*} \eta(b) V'(\eta(b)) \right] &= \sum_{b \in \overline{\Lambda}^*} \nabla \psi(b) E^{\mu_u} [V'(\eta(b))] + 2|\Lambda| - 2 \\ &\quad - 2E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} (\phi^{0,\eta}(x_b) - \psi(x_b)) V'(\eta(b)) \right]. \end{aligned}$$

Since we have

$$\lim_{\ell \rightarrow \infty} |\Lambda|^{-1} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*} \eta(b) V'(\eta(b)) \right] = 2 E^{\mu_u} \left[\sum_{i=1}^d \eta(e_i) V'(e_i) \right]$$

by translation-invariance of μ_u and also have

$$\lim_{\ell \rightarrow \infty} |\Lambda|^{-1} \sum_{b \in \overline{\Lambda}^*} \nabla \psi(b) E^{\mu_u} [V'(\eta(b))] = 2u \cdot \nabla \sigma(u)$$

by translation-invariance of μ_u and the identity (2.7), we obtain the conclusion once we have

$$\lim_{\ell \rightarrow \infty} \ell^{-d} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} (\phi^{0,\eta}(x_b) - \psi(x_b)) V'(\nabla \phi(b)) \right] = 0. \quad (4.1)$$

By Schwarz's inequality, we have

$$\begin{aligned} & \left| \ell^{-d} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} (\phi^{0,\eta}(x_b) - \psi(x_b)) V'(\nabla \phi(b)) \right] \right| \\ & \leq \gamma \ell^{-d} \sup_{x \in \partial \Lambda} E^{\mu_u} [|\phi^{0,\eta}(x) - \psi(x)|^2] + \gamma^{-1} \ell^{-d} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} |V'(\nabla \phi(b))|^2 \right] \\ & \leq \gamma \ell^{-d} \sup_{x \in \partial \Lambda} E^{\mu_u} [|\phi^{0,\eta}(x) - \psi(x)|^2] + \gamma^{-1} \ell^{-d} C_+^2 |\partial \Lambda| E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b = 0} \nabla \phi(b)^2 \right] \end{aligned}$$

for an arbitrary $\gamma > 0$. Let us estimate the first term in the right hand side. Let us take $\epsilon > 0$ arbitrarily. We can then take $\ell_0 \geq 1$ such that (3.9) with $t = 0$ holds for every $\ell \geq \ell_0$. Choosing γ as $\gamma = \ell^{-1} \epsilon^{-1}$, we obtain

$$\begin{aligned} & \left| \ell^{-d} E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b \notin \Lambda} (\phi^{0,\eta}(x_b) - \psi(x_b)) V'(\nabla \phi(b)) \right] \right| \\ & \leq \epsilon \ell^{-d+1} + \epsilon \ell^{-d+1} C_+^2 |\partial \Lambda| E^{\mu_u} \left[\sum_{b \in \overline{\Lambda}^*, x_b = 0} \nabla \phi(b)^2 \right], \end{aligned}$$

which shows (4.1) since $\ell^{-d+1} |\partial \Lambda|$ is bounded uniformly in ℓ . \square

Finally, we shall establish similar decomposition for $\nabla \sigma$ as in Section 3.3 of [12]. Applying the arguments there, we can obtain the uniform L^p -bound with $p > 2$ and the oscillation inequality for the discrete version of (2.11).

Proposition 4.3. *There exist a \mathbb{R}^d -valued function $a(u) = (a_i(u))_{1 \leq i \leq d} \in L^\infty(\mathbb{R}^d)^d$ and a matrix-valued function $A(u) = (A_{ij}(u))_{1 \leq i, j \leq d}$ satisfying*

$$c_- \mathbb{I} \leq A(u) \leq c_+ \mathbb{I}, \quad u \in \mathbb{R}^d \quad (4.2)$$

such that the identity

$$\nabla\sigma(u) = A(u)u + a(u), \quad u \in \mathbb{R}^d \quad (4.3)$$

holds.

Proof. We at first recall the relationship between the surface tension σ and the Gibbs measures:

$$\nabla_i\sigma(u) = E^{\mu_u}[V'(\eta(e_i))], \quad 1 \leq i \leq d,$$

where μ_u is the ergodic Gibbs measure with mean $u \in \mathbb{R}^d$. Using V_0 and g in Assumption 2.1, we shall take $A(u)$ as

$$A_{ij}(u) = E^{\mu_u} \left[\int_0^1 V_0''(\eta(e_i) - \lambda u_i) d\lambda \right] \delta_{ij}, \quad 1 \leq i, j \leq d$$

and $a(u)$ as

$$a_i(u) = E^{\mu_u} [V_0'(\eta(e_i) - u_i)] + E^{\mu_u} [g'(\eta(e_i))], \quad 1 \leq i \leq d.$$

It is easy to verify (4.3) and (4.2) with $A(u)$ and $a(u)$ defined above by using Assumption 2.1. Furthermore, the property $a(u) \in L^\infty(\mathbb{R}^d)^d$ is an immediate consequence of Theorem 3.2 and Schwarz's inequality. \square

4.3. Derivation of the macroscopic equation. We shall at first summarize the properties satisfied by the solution \bar{h}^N for the discretized PDE introduced in Section 3.1 of [12]. Since we have assumed the strict convexity of σ at Assumption 2.3, we obtain a priori bounds for \bar{h}^N in $C([0, T], L^2(D))$ and $L^2([0, T], H^1(D))$, see Proposition 3.1, Corollary 3.2 in [12]. Furthermore, since we have (4.3), we obtain the uniform bound of $\nabla \bar{h}^N$ in $L^p([0, T] \times D)$ and the oscillation inequality, see Propositions 3.3 and 3.4 in [12].

We next verify the coupled local equilibrium. Applying Theorem 3.1, Proposition 4.1 and the bound for \bar{h}^N stated above, we see that Proposition 4.2 in [12] is still valid. Summarizing above, we obtain that the arguments in Section 4.4 works completely, and we can finally conclude Theorem 2.1.

5. PROOF OF THEOREM 3.1

Our proof follows the argument of [5] however special care is required in view of the non ellipticity of the generator \mathcal{L} .

5.1. Generator and Dirichlet form. In this section, we mainly discuss properties of stationary measures of (2.4) while working on the general assumption, Assumption 2.1.

Since the calculation is based on the generator and the Dirichlet form, we shall introduce them before starting discussion.

We recall that the infinitesimal generator of (2.4) is given by

$$\mathcal{L}^{\mathbb{Z}^d} = \sum_{x \in \mathbb{Z}^d} \mathcal{L}_x, \quad (5.1)$$

where

$$\mathcal{L}_x = \sum_{b, b' \in (\mathbb{Z}^d)^*: x_b = x_{b'} = x} \left\{ 4 \frac{\partial^2}{\partial \eta(b) \partial \eta(b')} - 2V'(\eta(b)) \frac{\partial}{\partial \eta(b')} \right\}.$$

We also recall that we sometimes denote $\mathcal{L}^{\mathbb{Z}^d}$ by \mathcal{L} for simplicity.

We also introduce the finite version of (5.1). We at first introduce the state space for that. We define $\mathcal{X}_\Lambda \subset \mathbb{R}^{\Lambda^*}$ for a finite set $\Lambda \subset \mathbb{Z}^d$ by

$$\mathcal{X}_\Lambda = \{\eta \equiv \nabla \phi \in \mathbb{R}^{\Lambda^*}; \phi \in \mathbb{R}^\Lambda\}.$$

Note that $\mathcal{X}_{\Lambda,\xi}$ introduced in Section 2.2 and \mathcal{X}_Λ are state spaces for the dynamics with the boundary condition given by ξ and free boundary condition, respectively. For a finite set $\Lambda \subset \mathbb{Z}^d$ we define the differential operator \mathcal{L}^Λ and $\mathcal{L}^{\Lambda,f}$ by

$$\mathcal{L}^\Lambda = \sum_{x \in \Lambda} \mathcal{L}_x \quad (5.2)$$

and

$$\mathcal{L}^{\Lambda,f} = \sum_{x \in \Lambda} \mathcal{L}_x^{\Lambda,f}, \quad (5.3)$$

respectively. Here, $\mathcal{L}_x^{\Lambda,f}$ is the operator defined by

$$\mathcal{L}_x^{\Lambda,f} = \sum_{b,b' \in \Lambda^*: x_b = x_{b'} = x} \left\{ 4 \frac{\partial^2}{\partial \eta(b) \partial \eta(b')} - 2V'(\eta(b)) \frac{\partial}{\partial \eta(b')} \right\}.$$

The former is the generator associated to the dynamics η_t^Λ on $\mathcal{X}_{\Lambda,\xi}$ for given $\xi \in \mathcal{X}$, which is governed by SDEs

$$\begin{cases} d\eta_t^\Lambda(b) = -\nabla U_\cdot(\eta_t^\Lambda)(b) dt + \sqrt{2}dw_t(b), & b \in \overline{\Lambda^*}, \\ \eta_t^\Lambda(b) \equiv \xi(b), & b \in (\mathbb{Z}^d)^* \setminus \overline{\Lambda^*}. \end{cases}$$

The dynamics η_t^Λ is reversible under the finite volume Gibbs measure $\mu_{\Lambda,\xi}$ introduced in Section 2.2. The latter is the generator associated to $\eta_t^{\Lambda,f}$ governed by SDEs

$$d\eta_t^{\Lambda,f}(b) = -\nabla U_\cdot(\eta_t^{\Lambda,f})(b) dt + \sqrt{2}dw_t(b), \quad b \in \Lambda^*,$$

on \mathcal{X}_{Λ^*} , which corresponds to the dynamics (2.4) with free boundary condition. The character “f” in notions means free boundary condition. Here, H^Λ and U_x^Λ are defined by

$$\begin{aligned} H^\Lambda(\eta) &= \sum_{b \in \Lambda^*} V(\eta(b)), \\ U_x^\Lambda(\eta) &= \sum_{b \in \Lambda^*: x_b = x} V'(\eta(b)) \end{aligned}$$

for $\eta \in \mathcal{X}_{\Lambda^*}$. Note that $\eta_t^{\Lambda,f}$ is also reversible and the reversible measure is

$$\mu_{\Lambda,f}(d\eta) = Z_{\Lambda,f}^{-1} \exp(-H^\Lambda(\eta)) d\eta_{\Lambda^*},$$

on \mathcal{X}_{Λ^*} , where $d\eta_{\Lambda^*}$ is the Lebesgue measure on \mathcal{X}_{Λ^*} and $Z_{\Lambda,f}$ is the normalizing constant. The Dirichlet form associated to $\eta_t^{\Lambda,f}$ is given by

$$\mathcal{E}^{\Lambda,f}(f, g) = \int \sum_{x \in \Lambda} \left(\sum_{b \in \Lambda^*: x_b = x} \frac{\partial f}{\partial \eta(b)} \right) \left(\sum_{b \in \Lambda^*: x_b = x} \frac{\partial g}{\partial \eta(b)} \right) \mu_{\Lambda,f}(d\eta),$$

which plays a key role in the proof of the main theorem in this section, Theorem 3.1.

5.2. Proof of Theorem 3.1. In this subsection, let us complete the proof of Theorem 3.1, which is based on the method of [5]. Main tool is the integration-by-parts formula for $\mathcal{L}^{\Lambda, f}$ and entropy production rate. For the computation, we introduce a small lemma:

Lemma 5.1. *Let Λ be a finite subset of \mathbb{Z}^d .*

- (1) *Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a probability density on \mathbb{R} . Then, the image measure of $p(\phi(0)) \prod_{x \in \Lambda} d\phi(x)$ by the discrete gradient ∇ is nothing but the Lebesgue measure on \mathcal{X}_{Λ^*} .*
- (2) *If $f : \mathbb{R}^{\Lambda} \rightarrow \mathbb{R}$ is the form $f(\phi) = F(\nabla\phi)$, then*

$$\frac{\partial f}{\partial \phi(x)} = 2 \sum_{b: x_b = x} \frac{\partial F}{\partial \eta(b)}(\nabla\phi)$$

holds. Especially, if F is \mathcal{F}_{Λ^} -measurable, we have*

$$\frac{\partial f}{\partial \phi(x)} = 2 \sum_{b \in \Lambda^*: x_b = x} \frac{\partial F}{\partial \eta(b)}(\nabla\phi)$$

Proof. It is easy to see that

$$\int F(\nabla\phi) \delta_u(d\phi(0)) \prod_{x \in \Lambda \setminus \{0\}} d\phi(x) = \int F(\nabla\phi) \delta_v(d\phi(0)) \prod_{x \in \Lambda \setminus \{0\}} d\phi(x)$$

for every $u, v \in \mathbb{R}$ and bounded $F : \mathcal{X}_{\Lambda^*} \rightarrow \mathbb{R}$, which indicates that the integral

$$\int F(\nabla\phi) p(d\phi(0)) \prod_{x \in \Lambda} d\phi(x)$$

does not depend in the choice of a probability density p . Now, we check that the image measure has uniformity in \mathcal{X}_{Λ} . For $\xi \in \mathcal{X}_{\Lambda^*}$, there exists the $\psi \in \mathbb{R}^{\Lambda}$ such that $\psi(0) = 0$ and

$$\xi(b) = \nabla\psi(b)$$

holds. For a bounded function $F : \mathcal{X}_{\Lambda^*} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \int F(\nabla\phi + \xi) p(\phi(0)) \prod_{x \in \Lambda} d\phi(x) &= \int F(\nabla(\phi + \psi)) p(\phi(0)) \prod_{x \in \Lambda} d\phi(x) \\ &= \int F(\nabla\phi) p(\phi(0)) \prod_{x \in \Lambda} d\phi(x), \end{aligned}$$

which shows the first assertion.

For $F = F(\nabla\phi)$, we obtain

$$\frac{\partial F}{\partial \phi(x)} = \sum_{b: x_b = x} \frac{\partial F}{\partial \eta(b)} - \sum_{b: y_b = x} \frac{\partial F}{\partial \eta(b)} = 2 \sum_{b: x_b = x} \frac{\partial F}{\partial \eta(b)},$$

which shows the second assertion. □

Let us start to prove Theorem 3.1. We at first introduce $\Phi_{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi_{\lambda}(u) = \frac{\lambda}{a} (1 + (\lambda u)^2)^{-m},$$

where

$$a = \int_{\mathbb{R}} (1 + u^2)^{-m} du.$$

For $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$ we define $\Phi_n^\lambda : \mathcal{X}_{\Lambda_n^*} \rightarrow \mathbb{R}$ by

$$\Phi_n^\lambda(\eta) = \prod_{x \in \Lambda_n} \Phi_\lambda(\phi^{\eta,0}(x)),$$

where $\phi^{\eta,a}$ is the height variable satisfying $\nabla \phi^{\eta,a} = \eta$ and $\phi^{\eta,a}(0) = a$. Note that $\phi^{\eta,a}$ is uniquely determined by η and a . We also define $p_n^\lambda(\eta)$ by

$$p_n^\lambda(\eta) = \int \Phi_n^\lambda(\eta - \xi) \mu(d\xi).$$

Applying Lemma 5.1, we can easily verify that $p_n^\lambda(\eta)$ is probability density on $\mathbb{R}^{\mathcal{X}_{\Lambda_n^*}}$. Let $\Psi_n^\lambda(\eta, \xi) = \Phi_n^\lambda(\xi - \eta)$. Since $\Psi_n^\lambda(\cdot, \xi) \in C_{\text{loc}}^2(\mathcal{X})$, we have

$$\int \mathcal{L} \Psi_n^\lambda(\cdot, \xi)(\eta) \mu(d\eta) = 0.$$

Multiplying $F(\xi) \in C_{\text{loc}}^2(\mathcal{X})$ whose support is in Λ_n^* , and integrating in ξ by the uniform measure on $\mathcal{X}_{\Lambda_n^*}$, we obtain

$$\iint F(\xi) \mathcal{L} \Psi_n^\lambda(\cdot, \xi)(\eta) \mu(d\eta) d\xi_{\Lambda_n^*} = 0. \quad (5.4)$$

Applying Lemma 5.1, the right hand side is calculated as follows:

$$\iint F(\nabla \psi) \sum_{x \in \mathbb{Z}^d} \left(\frac{\partial^2 \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)^2} + \left(\sum_{b \in (\mathbb{Z}^d)^* : x_b = x} V'(\eta(b)) \right) \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)} \right) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta),$$

where $\nu_{\Lambda_n, p}$ is the measure on \mathbb{R}^{Λ_n} defined by

$$\nu_{\Lambda_n, p}(d\psi) = p(\psi(0)) \prod_{x \in \Lambda_n} d\psi(x)$$

with a probability density p on \mathbb{R} . Here, we have used the relationship

$$\begin{aligned} \frac{\partial \Psi_n^\lambda(\nabla \cdot, \nabla \psi)}{\partial \phi(x)}(\phi) &= -\frac{\partial \Phi_n^\lambda}{\partial \phi(x)}(\nabla \psi - \nabla \phi) = -\frac{\partial \Psi_n^\lambda(\nabla \phi, \nabla \cdot)}{\partial \psi(x)}(\psi) \\ \frac{\partial^2 \Psi_n^\lambda(\nabla \cdot, \nabla \psi)}{\partial \phi(x)^2}(\phi) &= \frac{\partial^2 \Phi_n^\lambda}{\partial \phi(x)^2}(\nabla \psi - \nabla \phi) = \frac{\partial^2 \Psi_n^\lambda(\nabla \phi, \nabla \cdot)}{\partial \psi(x)^2}(\psi) \end{aligned}$$

for $x \in \mathbb{Z}^d$ by the symmetry of Φ^λ . Noting

$$\frac{\partial \Phi_n^\lambda}{\partial \phi(x)} \equiv 0, \quad x \in \Lambda_n^c,$$

we obtain that the right hand side of (5.4) is computed as follows:

$$\begin{aligned} & \iint F(\nabla\psi) \sum_{x \in \Lambda_n} \frac{\partial^2 \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)^2} \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & + \iint F(\nabla\psi) \sum_{x \in \Lambda_n} \left(\sum_{b \in (\mathbb{Z}^d)^*: x_b = x} V'(\eta(b)) \right) \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)} \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & =: I_1 + I_2. \end{aligned}$$

We shall first calculate I_1 . Performing integration-by-parts in ψ , we have

$$\begin{aligned} I_1 = & - \iint \sum_{x \in \Lambda_n} \frac{\partial F(\nabla \cdot)}{\partial \psi(x)} \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)} \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & - \iint F(\nabla\psi) \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(0)} p'(\psi(0)) \prod_{x \in \Lambda_n} d\psi(x) \mu(d\eta). \end{aligned} \quad (5.5)$$

Noting that integrands of I_1 and the first term in the right hand side of (5.5) are function of $\nabla\psi$, each integral does not depend on the choice of p by Lemma 5.1 and therefore the second term does not also. On the other hand, since the second term converges to zero if taking the limit $p \rightarrow 0$ with $p' \rightarrow 0$, we conclude that the second term must be zero.

Let us choose F as

$$F(\nabla\psi) = f \left(\frac{p_n^\lambda(\nabla\psi)}{q_n(\nabla\psi)} \right), \quad (5.6)$$

with some bounded smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ and

$$q_n(\eta) = \exp(-H^\Lambda(\eta)), \quad \eta \in \mathcal{X}_{\Lambda_n^*}.$$

Noting

$$\frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \psi(x)} = \int \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)} \mu(d\eta),$$

we have

$$\begin{aligned} I_1 = & - \sum_{x \in \Lambda_n} \int f' \left(\frac{p_n^\lambda(\nabla\psi)}{q_n(\nabla\psi)} \right) \left(\frac{\partial}{\partial \psi(x)} \left(\frac{p_n^\lambda(\nabla \cdot)}{q_n(\nabla \cdot)} \right) \right)^2 q_n(\nabla\psi) \nu_{\Lambda_n, p}(d\psi) \\ & + \sum_{x \in \Lambda_n} \int f' \left(\frac{p_n^\lambda(\nabla\psi)}{q_n(\nabla\psi)} \right) U_x^\Lambda(\nabla\psi) p_n^\lambda(\nabla\psi) \nu_{\Lambda_n, p}(d\psi). \end{aligned} \quad (5.7)$$

Next, we shall compute I_2 . Performing the integration-by-parts in $\psi(x)$ again, we have

$$\begin{aligned} I_2 = & - \sum_{x \in \Lambda_n} \iint \frac{\partial F(\nabla\psi)}{\partial \psi(x)} U_x(\eta) \Psi_n^\lambda(\eta, \nabla\psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ = & - \sum_{x \in \Lambda_n} \iint \frac{\partial F(\nabla\psi)}{\partial \psi(x)} (U_x(\eta) - U_x^\Lambda(\eta)) \Psi_n^\lambda(\eta, \nabla\psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & - \sum_{x \in \Lambda_n} \iint \frac{\partial F(\nabla\psi)}{\partial \psi(x)} (U_x^\Lambda(\eta) - U_x^\Lambda(\nabla\psi)) \Psi_n^\lambda(\eta, \nabla\psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \end{aligned}$$

$$- \sum_{x \in \Lambda_n} \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} U_x^\Lambda(\nabla \psi) p_n^\lambda(\nabla \psi) \nu_{\Lambda_n, p}(d\psi).$$

Summarizing (5.4), (5.7) and above, we obtain

$$\begin{aligned} \sum_{x \in \Lambda_n} F_x^\lambda(n, f) &= - \sum_{x \in \Lambda_n} \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} (U_x(\eta) - U_x^\Lambda(\eta)) \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \quad (5.8) \\ &\quad - \sum_{x \in \Lambda_n} \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} (U_x^\Lambda(\eta) - U_x^\Lambda(\nabla \psi)) \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ &=: \sum_{x \in \Lambda_n} R_{1,x}^\lambda(n, f) + \sum_{x \in \Lambda_n} R_{2,x}^\lambda(n, f) \end{aligned}$$

if we take F as in (5.6), where $F_x^\lambda(n, f)$ is defined by

$$F_x^\lambda(n, f) := \int f' \left(\frac{p_n^\lambda(\nabla \phi)}{q_n(\nabla \phi)} \right) \left(\frac{\partial}{\partial \phi(x)} \left(\frac{p_n^\lambda(\nabla \cdot)}{q_n(\nabla \cdot)} \right) \right)^2 q_n(\nabla \phi) \nu_{n,p}(d\phi) \quad (5.9)$$

We note that if we can take $f(u) = \log u$, the left hand side coincides with the entropy production rate, that is,

$$\sum_{x \in \Lambda_n} F_x^\lambda(n, f) = \mathcal{E}^{\Lambda, f} \left(\sqrt{r_n^\lambda}, \sqrt{r_n^\lambda} \right) \quad (5.10)$$

holds, where r_n is the probability density with respect to $\mu^{\Lambda, f}$ given by

$$r_n^\lambda = Z_{\Lambda, f} p_n^\lambda q_n^{-1}.$$

In this case, we simply denote $F_x^\lambda(n, f)$ by $F_x^\lambda(n)$. Before continuing the discussion for (5.8), we shall verify the integrability of integrands in (5.9) with $f(u) = \log u$.

Lemma 5.2. *For every n, λ, x , the integral (5.9) is finite if $f(x) = \log x$.*

Proof. We at first note that

$$F_x^\lambda(n) = \int \left((p_n^\lambda)^{-1} \frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \phi(x)} - q_n^{-1} \frac{\partial q_n(\nabla \cdot)}{\partial \phi(x)} \right)^2 p_n^\lambda(\phi) \nu_{n,p}(d\phi).$$

By the definition of Φ_λ , we have for $x \neq 0$

$$\frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \phi(x)} = \int (-2m\lambda^2) \frac{\phi(x) - \phi(0) - \phi^{\xi, 0}(x)}{1 + \lambda^2(\phi(x) - \phi(0) - \phi^{\xi, 0}(x))^2} \Phi_n^\lambda(\xi - \nabla \phi) \mu(d\xi)$$

and

$$\left| \frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \phi(x)} \right| \leq m\lambda \int \Phi_n^\lambda(\xi - \nabla \phi) \mu(d\xi) = m\lambda p_n^\lambda(\nabla \phi).$$

Here, we have used

$$\max_{u \in \mathbb{R}} \left| \frac{u}{1 + \lambda^2 u^2} \right| = \frac{1}{2\lambda}.$$

On the other hand, we have for $x = 0$

$$\frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \phi(0)} = \int 2m\lambda^2 \sum_{x \in \Lambda_n} \frac{\phi(x) - \phi(0) - \phi^{\xi,0}(x)}{1 + \lambda^2(\phi(x) - \phi(0) - \phi^{\xi,0}(x))^2} \Phi_n^\lambda(\xi - \nabla \phi) \mu(d\xi)$$

and

$$\left| \frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \phi(0)} \right| \leq m\lambda |\Lambda_n| \int \Phi_n^\lambda(\xi - \nabla \phi) \mu(d\xi) = m\lambda |\Lambda_n| p_n^\lambda(\nabla \phi).$$

We conclude that $(p_n^\lambda)^{-1} \partial p_n^\lambda(\nabla \cdot) / \partial \phi(x)$ is square-integrable.

Next, let us verify that $(q_n)^{-1} \partial q_n(\nabla \cdot) / \partial \phi(x)$ is also square-integrable. Note that

$$U_x^{\Lambda_n}(\nabla \phi) = q_n^{-1} \frac{\partial q_n(\nabla \cdot)}{\partial \phi(x)},$$

and

$$\begin{aligned} & \int U_x^{\Lambda_n}(\nabla \phi)^2 p_n^\lambda(\nabla \phi) \nu_{n,p}(d\phi) \\ & \leq 2 \iint U_x^{\Lambda_n}(\xi)^2 \Phi_n^\lambda(\nabla \phi - \xi) \nu_{n,p}(d\phi) \mu(d\xi) \\ & \quad + 2 \iint (U_x^{\Lambda_n}(\xi) - U_x^{\Lambda_n}(\nabla \phi))^2 \Phi_n^\lambda(\nabla \phi - \xi) \nu_{n,p}(d\phi) \mu(d\xi) \\ & \leq 2 \int b_x(\xi, n)^2 \mu(d\xi) \\ & \quad + K \iint \sum_{b \in \Lambda_n^*: x_b = x} |\xi(b) - \nabla \phi(b)|^2 \Phi_n^\lambda(\nabla \phi - \xi) \nu_{n,p}(d\phi) \mu(d\xi) \end{aligned}$$

for some $K > 0$ from Lipschitz continuity of V' . It is easy to see that the first term is finite by using temperedness of μ . We can obtain that the second term is also finite since we have

$$\begin{aligned} & \iint |\xi(b) - \nabla \phi(b)|^2 \Phi_n^\lambda(\nabla \phi - \xi) \nu_{n,p}(d\phi) \mu(d\xi) \\ & \leq 2 \iint |\phi^{\xi,0}(x_b) - \phi(x_b) + \phi(0)|^2 \Phi_n^\lambda(\nabla \phi - \xi) \nu_{n,p}(d\phi) \mu(d\xi) \\ & \quad + 2 \iint |\phi^{\xi,0}(y_b) - \phi(y_b) + \phi(0)|^2 \Phi_n^\lambda(\nabla \phi - \xi) \nu_{n,p}(d\phi) \mu(d\xi). \end{aligned} \tag{5.11}$$

and

$$\int_{\mathbb{R}} (x - a)^2 \Phi_\lambda(x - a) dx \leq C\lambda^{-2} \tag{5.12}$$

by the definition of Φ . □

Using $F(n)$, we can now bound the right hand side of (5.8):

Lemma 5.3. *Assume that the function f satisfies $0 \leq uf'(u) \leq 1$ for every $u > 0$. We then have bounds for $R_{1,x}^\lambda(n, f)$ and $R_{2,x}^\lambda(n, f)$ in (5.8) as follows:*

$$|R_{1,x}^\lambda(n, f)| \leq K_1 C_x(n)^{1/2} F_x^\lambda(n)^{1/2} \quad (5.13)$$

$$|R_{2,x}^\lambda(n, f)| \leq K_2 \lambda^{-1} F_x^\lambda(n)^{1/2} \quad (5.14)$$

with some constants $K_1, K_2 > 0$ independent in n and λ , where $C_x(n)$ is defined by

$$C_x(n) = \sum_{b \in (\mathbb{Z}^d)^* \setminus \Lambda^*: x_b = x} \int c_b^2(\eta, n, \mu) \mu(d\eta),$$

$$c_b(\eta, n, \mu) = \int V'(\eta(b)) \mu(d\eta | \mathcal{F}_{(\Lambda_n^*)^c})(\eta).$$

Proof. We at first obtain

$$\begin{aligned} & \iint f' \left(\frac{p_n^\lambda}{q_n} \right)^2 \left(\frac{\partial}{\partial \psi(x)} \left(\frac{p_n^\lambda}{q_n} \right) \right)^2 \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & \leq \iint \left(\frac{p_n^\lambda}{q_n} \right)^{-1} \left(\frac{\partial}{\partial \psi(x)} \left(\frac{p_n^\lambda}{q_n} \right) \right)^2 q_n(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) = F_x^\lambda(n) \end{aligned}$$

by the assumption on f . We therefore get

$$\begin{aligned} |R_{1,x}^\lambda(n, f)| & \leq F_x^\lambda(n)^{1/2} \left(\iint (U_x(\eta) - U_x^\Lambda(\eta))^2 \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \right)^{1/2} \\ & \leq K F_x^\lambda(n)^{1/2} \left(\int \sum_{b \in (\mathbb{Z}^d)^* \setminus \Lambda^*: x_b = x} c_b^2(\eta, n, \mu) \mu(d\eta) \right)^{1/2} \end{aligned}$$

for some constant $K_1 > 0$, which shows (5.13). We note that $R_{1,x}^\lambda(n)$ is equal to zero if x is not on the boundary of Λ_n . Also for (5.14), we obtain

$$\begin{aligned} |R_{2,x}^\lambda(n, f)| & \leq F_x^\lambda(n)^{1/2} \left(\iint (U_x^\Lambda(\eta) - U_x^\Lambda(\nabla \psi))^2 \Phi_n^\lambda(\eta - \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \right)^{1/2} \\ & \leq K_2 \lambda^{-1} F_x^\lambda(n)^{1/2} \end{aligned}$$

for some constant $K_2 > 0$ by applying (5.11) and (5.12) again. \square

Summarizing above and applying Schwarz's inequality, we obtain

$$\begin{aligned} \sum_{x \in \Lambda_n} F_x^\lambda(n, f) & \leq \sum_{x \in \Lambda_n} (F_x^\lambda(n))^{1/2} (K_1 C_x(n)^{1/2} + K_2 \lambda^{-1}) \\ & \leq \frac{1}{2} \sum_{x \in \Lambda_n} F_x^\lambda(n) + K_1^2 \sum_{x \in \Lambda_n} C_x(n) + K_2^2 \lambda^{-2}. \end{aligned}$$

By taking limit $f(u)$ to $\log u$ with keeping $0 \leq uf'(u) \leq 1$ and applying Fatou's lemma.

$$\sum_{x \in \Lambda_n} F_x^\lambda(n) \leq 2K_1^2 \sum_{x \in \Lambda_n} C_x(n) + 2K_2^2 \lambda^{-2} |\Lambda_n|. \quad (5.15)$$

Here, using Jensen's inequality and shift-invariance and temperedness of μ , we get

$$\int c_b(\xi, n, \mu)^2 \mu(d\xi) \leq K < \infty,$$

with a constant $K > 0$ independent of n and b , and therefore get

$$\sum_{x \in \Lambda_n} C_x(n) \leq 2dK|\Lambda_n \setminus \Lambda_{n-1}|. \quad (5.16)$$

Summarizing (5.15) and (5.16), we get

$$\sum_{x \in \Lambda_n} F_x^\lambda(n) \leq 4dK_1^2 K |\Lambda_n \setminus \Lambda_{n-1}| + 2K_2^2 \lambda^{-2} |\Lambda_n|. \quad (5.17)$$

We note that the left hand side of (5.17) coincides with the entropy production rate, that is, the identity

$$\begin{aligned} \sum_{x \in \Lambda_n} F_x^\lambda(n) &= \mathcal{E}^{\Lambda_n, f}(\sqrt{r_n^\lambda}, \sqrt{r_n^\lambda}) \\ &= \sup \left\{ \int_{\mathcal{X}_{\Lambda_n^*}} \frac{-\mathcal{L}^{\Lambda_n, f} u}{u} d\mu_n^\lambda; u \in C_b^2(\mathcal{X}), \mathcal{F}_{\Lambda_n^*}\text{-measurable}, u \geq 1 \right\} \end{aligned}$$

holds, where the probability measure μ_n^λ on Λ_n^* is defined by $d\mu_n^\lambda = r_n^\lambda d\mu^{\Lambda_n, f}$. We note that the core of Dirichlet form $\mathcal{E}^{\Lambda_n, f}$ is the family of smooth $\mathcal{F}_{\Lambda_n^*}$ -measurable functions.

For $\ell \in \mathbb{N}$, let us take $\tilde{\Lambda} \subset \Lambda_n$ by

$$\tilde{\Lambda} = \bigcup_{x \in ((2\ell+3)\mathbb{Z})^d; \Lambda_\ell(x) \subset \Lambda_{n-1}} \Lambda_\ell(x),$$

where $\Lambda_\ell(x) = \Lambda_\ell + x$. Because boxes $\Lambda_\ell(x)$ appearing above are disjoint, we get

$$\sum_{x \in ((2\ell+3)\mathbb{Z})^d; \Lambda_\ell(x) \subset \Lambda_{n-1}} \sum_{y \in \Lambda_\ell(x)} F_y^\lambda(n) \leq \sum_{x \in \Lambda_n} F_x^\lambda(n).$$

On the other hand, we have

$$\sum_{y \in \Lambda_\ell(x)} F_y^\lambda(n) = I^{\Lambda_\ell(x)}(\mu_n^\lambda),$$

where the right hand side is the entropy production rate defined by

$$I^\Lambda(\tilde{\mu}) := \sup \left\{ \int \frac{-\mathcal{L}^\Lambda u}{u} d\tilde{\mu}; u \in C_b^2(\mathcal{X}), \mathcal{F}_{\Lambda^*}\text{-measurable}, u \geq 1 \right\}$$

for a finite $\Lambda \subset \mathbb{Z}^d$ and $\tilde{\mu}$ on \mathcal{X} or $\mathcal{X}_{\Lambda_n^*}$ with n large enough. Repeating the argument as in the proof of Lemma 4.2 in [7], we obtain the Gibbsian property of μ .

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